Generalized Hooke's Law
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The purpose of this note is to describe the usefulness of generalized Hooke's Law, or

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}$$

which describes in shorthand notation three equations for normal strain

$$\varepsilon_{11} = \frac{1}{E}\sigma_{11} - \frac{\nu}{E}\sigma_{22} - \frac{\nu}{E}\sigma_{33}$$

$$\vdots$$

and three equations for shear strain

$$\varepsilon_{12} = \frac{1+\nu}{E}\sigma_{12}$$

$$\vdots$$

Generalized Hooke's Law applies in the case of three-dimensional loading of an *isotropic* material. It lies in an intermediate position between the simplest case of uniaxial loading of a bar (the scalar relationship $\sigma = E\varepsilon$) and general loading of anisotropic materials (the tensor relationship $\boldsymbol{\sigma} = \mathbb{C}\boldsymbol{\varepsilon}$, or $\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$). Generalized Hooke's Law is particularly useful for deriving *effective moduli* for various common stress states. These moduli (shear modulus, bulk modulus, biaxial modulus, and so on) are frequently discussed but less frequently derived; knowledge of generalized Hooke's Law makes their derivation simple. Several of these stress states are examined here.

1. *Uniaxial*. Consider the case of a bar (by definition, a long, thin object) under uniaxial loading along the *1*-axis. If the bar is sufficiently long and thin, we might assume that any lateral internal stresses σ_{22} and σ_{33} are zero, and thus obtain uniaxial Hooke's Law:

   $$\varepsilon_{11} = \frac{1}{E}\sigma_{11}$$

   There is also the expected lateral contraction associated with Poisson's ratio:

   $$\varepsilon_{22} = \varepsilon_{33} = -\frac{\nu}{E}\sigma_{11}$$

   All shear strains are zero.

2. *Pure shear*. In the case of pure shear (σ_{12} nonzero, all other stresses zero), we have

   $$\varepsilon_{12} = \frac{1+\nu}{E}\sigma_{12}$$

   and all other strains are zero.

The ratio of the shear stress to the engineering shear strain $\gamma_{ij} = 2\varepsilon_{ij}$ is the *shear modulus*

$$G = \frac{E}{2(1+\nu)}$$

(Note: the existence of two different shear strains is confusing, but each has its uses. The engineering shear strain γ conveniently describes the angular decrease (in radians) in a right angle under shear. The tensor shear strain ε makes shorthand equations like $\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ work for any i and j.)

3. *Hydrostatic*. Let us assume now that the stress state is hydrostatic (all shear stresses zero, normal stresses $\sigma_{11} = \sigma_{22} = \sigma_{33} = p$). The normal strain in any direction is

   $$\varepsilon = \frac{1-2\nu}{E}p$$

   while all shear strains are found to be zero.

   The normalized change in volume is

   $$\frac{\Delta V}{V} = (1+\varepsilon)^3 - 1 \approx (1+3\varepsilon) - 1 = 3\varepsilon = \frac{3(1-2\nu)}{E}p$$

   where we have assumed the strains to be small. The ratio of the pressure to the normalized change in volume is the *bulk modulus*

   $$K = \frac{E}{3(1-2\nu)}$$

4. *Equibiaxial plane stress*. Consider the case when the stress in one direction is zero (σ_{33}) and the other two stresses are equal ($\sigma_{11} = \sigma_{22}$). This is the case of equibiaxial plane stress: the stress is confined to one plane (here, the *1–2* plane), and the in-plane normal stress is independent of direction. The in-plane strain is found via generalized Hooke's Law to be

   $$\varepsilon_{11} = \frac{1}{E}\sigma_{11} - \frac{\nu}{E}\sigma_{22} = \frac{1-\nu}{E}\sigma_{11}$$

   The effective in-plane Young's modulus (the ratio of in-plane stress to in-plane strain) is termed the *biaxial modulus*

   $$M = \frac{E}{1-\nu}$$

   Note also that generalized Hooke's Law provides a value for ε_{33} in terms of the in-plane strain:

   $$\varepsilon_{33} = -\frac{2\nu}{E}\sigma_{11} = -\frac{2\nu}{1-\nu}\varepsilon_{11}$$

5. *Plane strain*. Finally, consider the case where the strain in one direction (say, the direction of the *3*-axis) is constrained to be zero. This is the case of *plane strain*: the strain is confined to one plane (here, the *1–2* plane). An example of this state is found in the bending of plates. If a plate is sufficiently wide in the *3*-direction, the contraction in this direction is negligible

during bending. For simplicity, let us assume that $\sigma_{22} = 0$ but that σ_{11} exists and is related to the bending stresses. By setting $\varepsilon_{33} = 0$, we use generalized Hooke's Law to obtain

$$\sigma_{33} = \nu\sigma_{11}$$

$$\varepsilon_{11} = \frac{1 - \nu^2}{E}\sigma_{11}$$

and find that the effective Young's modulus under the plane strain condition is

$$E' = \frac{E}{1 - \nu^2}$$

The bending stiffness of a sufficiently wide plate is thus slightly higher than that of a bar (10% higher for a Poisson's ratio of 0.3). In general, the application of a deformation constraint increases the effective stiffness of an object.

The following points may be useful:

- The relationships described above only applies to isotropic materials; anisotropic materials, in general, need to be analyzed via the full relationship $\boldsymbol{\sigma} = \mathbb{C}\boldsymbol{\varepsilon}$. But note some exceptions: polycrystalline materials can often be modeled as isotropic due to the averaging of a large number of independently oriented grains. Cubic materials in thin-film form can be modeled as isotropic in the plane of the film under certain conditions (e.g., {100} or {111} surface orientation).

- The simple relationship $\sigma = E\varepsilon$ applies only in the case of uniaxial loading of a bar, since the lateral internal stresses σ_{22} and σ_{33} may not be zero for less elongated objects. When considering uniaxial compression of a very wide, flat object, for example, we would not generally be able to assume that lateral stresses are zero; we would more properly assume that lateral *strains* are zero.

- Normal and shear stresses and strains are totally decoupled in isotropic materials. This is actually the case for many anisotropic materials as well; in the cubic, tetragonal, and hexagonal crystal types, for example, the coefficient C_{14} (among others) equals zero.[1] Recall that C_{14} is contracted notation for C_{1123}, which describes the relationship between normal stress σ_{11} and shear strain ε_{23}.

- Finally, the inverted version of generalized Hooke's Law, in which stress is defined in terms of strain, is occasionally useful and can be expressed as[2]

$$\sigma_{ij} = \frac{E}{1 + \nu}\varepsilon_{ij} + \frac{\nu E}{(1 + \nu)(1 - 2\nu)}\varepsilon_{kk}\delta_{ij}$$

[1] Nye, *Physical Properties of Crystals*, Oxford University Press: London, 1985.

[2] Ugural and Fenster, *Advanced Strength and Applied Elasticity*, Upper Saddle River, NJ: Prentice Hall, 1995.