Notes on true strain e vs. engineering strain ε
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- Strain is normalized deformation. We can express this relationship in differential form for a bar undergoing axial deformation as $d(\text{strain}) = \frac{dL}{L}$, or an infinitesimal deformation dL normalized to length L.

- We properly find the total axial strain e (known as the *true strain*) by integrating this expression from the initial length L_0 to the final length L_F:

$$e \equiv \int_{L_0}^{L_F} \frac{dL}{L} = \ln(L)\Big|_{L_0}^{L_F} = \ln\left(\frac{L_F}{L_0}\right) = \ln\left(\frac{L_0 + \Delta L}{L_0}\right) = \ln\left(1 + \frac{\Delta L}{L_0}\right)$$

  where ΔL is the amount of deformation (positive for elongation). The length L is kept inside the integral because it changes as the bar deforms from length L_0 to length L_F.

- If ΔL is small compared to the length of the bar, then $L \approx L_0$ at all stages of the deformation process and the $\frac{1}{L}$ term can be taken outside the integral. The resulting approximate strain ε is known as the *engineering strain*:

$$\varepsilon \equiv \frac{1}{L_0} \int_{L_0}^{L_F} dL = \frac{1}{L_0} \, L\Big|_{L_0}^{L_F} = \frac{L_F - L_0}{L_0} = \frac{\Delta L}{L_0}$$

- The same result is acquired through a Taylor series expansion of $e = \ln\left(1 + \frac{\Delta L}{L_0}\right)$. The Taylor series

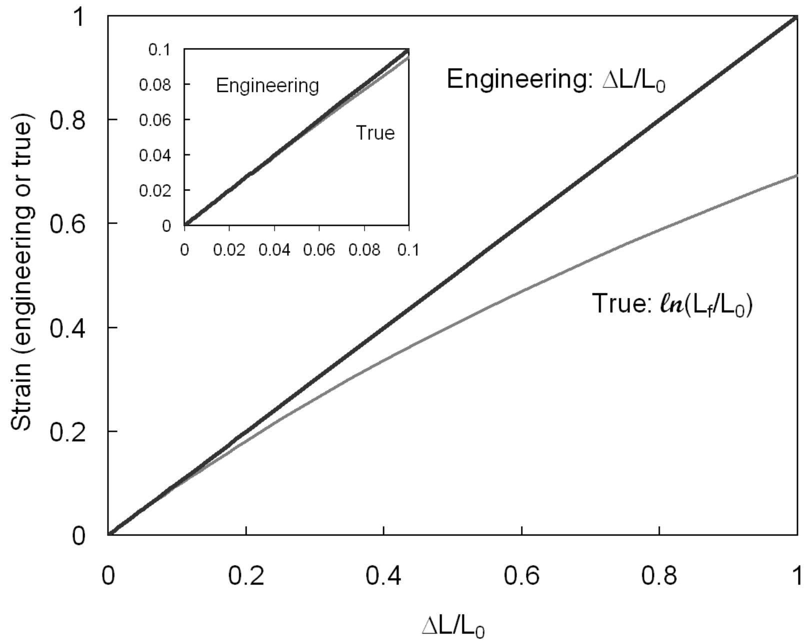
$$f(x_0 + \Delta x) = f(x_0) + \left.\frac{\partial f(x)}{\partial x}\right|_{x_0} \Delta x + \frac{1}{2!} \left.\frac{\partial^2 f(x)}{\partial x^2}\right|_{x_0} \Delta x^2 + \frac{1}{3!} \left.\frac{\partial^3 f(x)}{\partial x^3}\right|_{x_0} \Delta x^3 + \ldots$$

  is especially useful for estimating $f(x_0 + \Delta x)$ when $f(x_0)$ is known and Δx is small. In this case, $f(x) = \ln(x)$, $x_0 = 1$, and $\Delta x = \frac{\Delta L}{L_0}$. So we have

$$e = \ln\left(1 + \frac{\Delta L}{L_0}\right) \approx \ln(1) + (1)^{-1}\left(\frac{\Delta L}{L_0}\right) = \left(\frac{\Delta L}{L_0}\right) = \varepsilon$$

- The relationship between true strain e and engineering strain ε is exactly $e = \ln(1 + \varepsilon)$. Both strains, as normalized quantities, are unitless. Small strains are sometimes described as a percent (e.g., 0.2%), or in μ, or micros (i.e., 2000 μ = 2000 parts per million = 0.2%).

- How close are the the values of e and ε? Plotted below are the engineering and true strains for values up to 1. The agreement is quite good for strains of less than 0.1 (see inset). Note that, by convention, an engineering material is considered to have yielded—deformed beyond recovery—at an engineering strain of 0.002, or 0.2%. At this value, the difference between engineering and true strain is less than one part in a thousand.



- Characteristics of true strain e:
  1. It's the exact value, not an approximation.
  2. Sequential strains can be added: if two strains e_1 and e_2 are executed sequentially, the total strain is
  $$e_1 + e_2 = \ln\left(\frac{L_1}{L_0}\right) + \ln\left(\frac{L_2}{L_1}\right) = \ln\left(\frac{L_1}{L_0} \cdot \frac{L_2}{L_1}\right) = \ln\left(\frac{L_2}{L_0}\right)$$
  This is not the case with engineering strain, where the total strain is
  $$\frac{L_2 - L_0}{L_0} \neq \varepsilon_1 + \varepsilon_2 = \frac{L_1 - L_0}{L_0} + \frac{L_2 - L_1}{L_1}$$
  3. It's used to characterize materials that deform by large amounts (considerable fractions of their length up to many times their length). A quick look at the literature shows that true strain has been recently used to characterize materials like polyamide yarn, epoxy, rubber, and cartilage.
  4. It's geometrically symmetric: that is, if the strain associated with being stretched to n times the original length is e, then the strain associated with being compressed to $\frac{1}{n}$ the original length is $-e$.

- Characteristics of engineering strain ε:
  1. It's easier to calculate.
  2. It's overwhelmingly preferred in engineering analyses of materials that experience only small strains (including the common construction materials concrete, wood, and steel, for example, under normal use).
  3. It's symmetric in terms of displacements: that is, if the strain associated with being stretched a distance ΔL is ε, then the strain associated with being compressed a distance ΔL is $-\varepsilon$.